JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 6, No. 2, April–June 1992

Effect of Linear-Anisotropic Scattering on Spectral Emission from Cylindrical Plumes

S. T. Thynell*
Pennsylvania State University, University Park, Pennsylvania 16802

The corresponding integral form of the equation of radiative transfer in absorbing, emitting, linear-anisotropic scattering media is used to compute the spectral hemispherical and the spectral plume emissivities of a one-dimensional cylinder. The solution is based on using power series representations of the temperature and incident radiation profiles. The expansion coefficients for the temperature are assumed known, whereas the expansion coefficients for the incident radiation are solved for by the application of a collocation strategy based on the zeros of the Chebyshev polynomials. The method of solution is employed to study primarily the interaction between an absorbing/emitting gas phase and an absorbing/emitting/scattering particle phase. The numerical results indicate that the plume emissivity is enhanced by as much as 20% by the presence of purely scattering particles.

Nomenclature

A_i = expansion coefficients to Planck function
a = linear anisotropic scattering coefficient
B_i = expansion coefficients to $G(r)$
C = interaction parameter
$D_{m,n}(r)$ = functions defined in the Appendix
dA_1 = elemental surface area on plume
dA_2 = elemental surface area of detector
$dQ_{1\lambda}$ = flux from elemental strip
$d^2Q_{1\lambda}$ = flux from elemental surface dA_1
$d\Omega_{12}$ = solid angle
$E_{m,n}(r)$ = functions defined in the Appendix
$G(r)$ = incident radiation
H = height or length of plume
I = order of expansion of incident radiation
$I_b(T)$ = dimensional Planck function
$I_n(x)$ = modified Bessel function of the first kind
$I(r,\theta,\phi)$ = radiation intensity
$I^*(r,\theta,\phi)$ = $I(r,\theta,\phi)/I_b(T)$, normalized radiation intensity
J = order of expansion of Planck function
$K_n(x)$ = modified Bessel function of the second kind
$L_{m,n}(r,x)$ = kernels in integral equations
n = index of refraction
$\hat{n}$ = outward normal unit vector
$P(\Theta)$ = $\simeq 1 + a \cos \Theta$, scattering phase function
$q(r)$ = net radiation heat flux in r-direction
R = $(\sigma_s + \kappa_p + \kappa_g)R_{ph}$, optical radius of plume
R_g = $\kappa_g R_{ph}$, spectral optical radius of plume due to participating gases
R_p = $(\sigma_s + \kappa_p)R_{ph}$, spectral optical radius of plume due to particles
R_{ph} = physical radius of plume
r = optical radial variable
r_k = collocation point
S = distance between plume and detector
T_m = mean temperature of plume
$T(r)$ = temperature of medium
X_i^* = functions defined in the Appendix
$X_i(R,\theta,\phi)$ = functions defined in the Appendix
x = dummy variable of integration
Y_i^* = functions defined in the Appendix
$Y_i(R,\theta,\phi)$ = functions defined in the Appendix
β_1 = polar angle defined in Fig. 2
β_2 = polar angle defined in Fig. 2
ε_H = hemispherical emissivity
ε_λ = plume emissivity
θ = polar angle
κ_g = gas absorption coefficient
κ_p = particle absorption coefficient
λ = wavelength
σ_s = scattering coefficient
ϕ = azimuth angle
Ω = solid angle
ω = $\sigma_s/(\sigma_s + \kappa_p + \kappa_g)$, single scattering albedo
ω_p = $\sigma_s/(\sigma_s + \kappa_p)$, particle single scattering albedo

Introduction

THE analysis of IR signatures from rocket plumes has been the subject of numerous studies. The interest is related to several aspects of propulsion including, among others, the development of methods for improving the combustion efficiency, the determination of the effects of combustion instabilities, the prediction of radiative base heating, the simulation of plume signatures, and the design of protective measures from heat-seeking missiles. In addition, the development of new formulations of solid propellants, including more energetic binders and additives such as boron, have required not only continued efforts in advancing the predictive capability but also continuous updating of the database of computer codes describing the plume IR signature.

Numerous studies have dealt with examining the effects of particles on the plume signature and plume emission. Lyons et al.[1] used both a Monte Carlo simulation and the Aerodyne Radiation Code (ARC) to predict the IR signature for optically thin plumes, but they deduced that further studies of calculational methods as well as further measurements of particle properties are needed. Using the Standard Infrared Radiation Model (SIRRM),[2] Nelson[3,4] concluded that the plume signatures are sensitive to the particle size and the amount of aluminum oxide present, and that anisotropic scattering alters the IR signature to some extent. In a subsequent study, Nelson and Tucker[5] used the SIRRM code to deduce that the effect of B_2O_3 particles on the IR signature is significant, suggesting further studies should be performed on determin-

Presented as Paper 90-0137 at the AIAA 28th Aerospace Sciences Meeting, Reno, NV, Jan. 8–11, 1990; received Oct. 24, 1990; revision received Feb. 5, 1991; accepted for publication Feb. 9, 1991.

*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

ing the complex index of refraction of such particles. Stockham and Love[6] employed the Monte Carlo method to investigate the thermal radiation to the exterior base region of a finite cylindrical dispersion of absorbing, emitting, and anisotropically scattering particles. Edwards, Sakurai, and Babikian[7] employed a hybrid Monte Carlo radiosity/irradiation technique based on isotropic scattering to further advance the Bobco model[8] for predicting the role of plume radiation to aft-end equipment. Young[9] attempted a numerical inversion of the integral form of the equation of transfer for obtaining temperature profiles of rocket exhaust plumes, but, as also noted by Ho and Özisik,[10] it is difficult to deduce accurately the radiative properties from such inversions. Recent studies in one-dimensional, plane-parallel[11-13] and cylindrical[14] media have indicated that the presence of isotropically scattering particles in mixtures of either H_2O or CO_2 alters the hemispherical and the directional emissivities significantly. It should be noted, however, the total hemispherical emissivity is almost always reduced due to the shielding effect of the particles.

Although the previous brief review indicates that there are numerous studies available in the literature, it is suggested that additional studies should be attempted in order to advance the solution methodology to radiation transfer and to ascertain the importance of anisotropic scattering on the directional emission from rocket plumes. There are two objectives of this work: 1) to present a highly accurate solution to the equation of transfer in absorbing, emitting, linear-anisotropic scattering, one-dimensional cylindrical media; and 2) to employ the solution for the analysis of the role of the particles in either enhancing or inhibiting the emission of radiant energy from the plume. The analysis is based on the assumptions that the temperatures of particle and gas phases are equal, and that the radiative properties are known.

Analysis

Formulation of Radiation Transfer

We consider radiative heat transfer in absorbing, emitting, linear-anisotropic scattering, one-dimensional cylindrical media. The spectral radiative properties are assumed spatially independent. The medium contains spatially varying radiant energy sources of intensity $I_{b\lambda}[T(r)]$, where $T(r)$ is the temperature of the particles and gases and $I_{b\lambda}(T)$ is the Planck function. The cylinder is surrounded by a cold and nonreflecting medium, and no radiant energy is incident on the cylinder from, for example, the sun. A schematic of the physical model and coordinates is illustrated in Fig. 1. As shown recently,[15] it is possible to recast the equation of transfer into a corresponding integral form of the Fredholm type. Such transformation is advantageous for two important reasons. First, it reduces the number of independent variables from three (r,θ,ϕ) to one (r), thus simplifying the computational effort. Second, a solution is constructed for dependent variables [$G(r)$, $q(r)$ or higher moments of the intensity] that are smooth functions (solid-angle integrated), whereas the radiation intensity often is singular and an accurate solution is more difficult to obtain. Omitting for sake of clarity the wavelength dependence, the integral form of the equation of transfer is represented by[15]

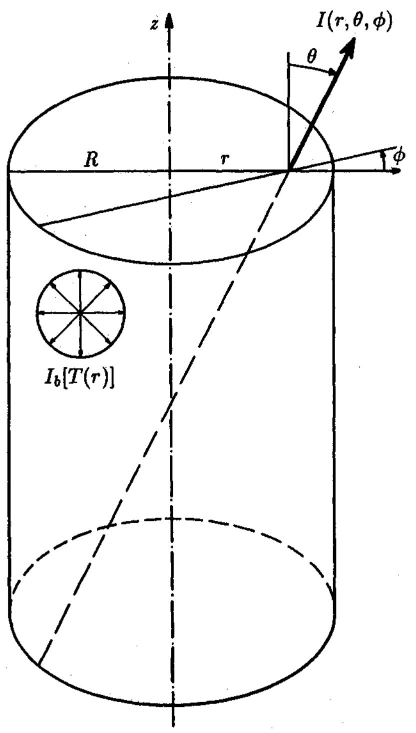

Fig. 1 Physical model and coordinates.

$$G(r) = \int_{x=0}^{R} \left[\left\{ (1-\omega) I_b[T(x)] + \frac{\omega}{4\pi} G(x) \right\} L_{0,0}(r,x) + a \frac{\omega}{4\pi} q(x) L_{0,1}(r,x) \right] x \, \mathrm{d}x \tag{1}$$

and

$$q(r) = \int_{x=0}^{R} \left[\left\{ (1-\omega) I_b[T(x)] + \frac{\omega}{4\pi} G(x) \right\} L_{1,0}(r,x) + a \frac{\omega}{4\pi} q(x) L_{1,1}(r,x) \right] x \, \mathrm{d}x \tag{2}$$

where the kernels $L_{m,n}(r,x)$ are defined by

$$L_{m,n}(r,x) = 4\pi \int_{\mu=0}^{1} K_m(r/\mu) I_n(x/\mu) \mu^{m+n-2} \, \mathrm{d}\mu \qquad x < r \tag{3a}$$

$$L_{m,n}(r,x) = 4\pi \int_{\mu=0}^{1} I_m(r/\mu) K_n(x/\mu) (-\mu)^{m+n-2} \, \mathrm{d}\mu \qquad x > r \tag{3b}$$

Here, the incident radiation $G(r)$ and the net radiant heat flux $q(r)$ are, respectively, defined by

$$G(r) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(r,\theta,\phi) \sin\theta \, \mathrm{d}\theta \, d\phi \tag{4a}$$

$$q(r) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(r,\theta,\phi) \sin^2\theta \cos\phi \, \mathrm{d}\theta \, d\phi \tag{4b}$$

In addition, $I(r,\theta,\phi)$ is the spectral radiation intensity, r is the spectral radial variable (radial optical path length), θ and ϕ are the polar and azimuthal angles, respectively, ω is the single scattering albedo, and $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of the first and second kind, respectively.

Method of Solution

To solve the coupled system of integral equations for the incident radiation and the net radiant heat flux, we first assume that the temperature is continuous throughout the plume and employ the following expansion:

$$I_b[T(r)] = \sum_{j=0}^{J} A_j r^{2j}, \qquad 0 \le r \le R \tag{5}$$

and

$$G(r) = \sum_{i=0}^{I} B_i r^{2i}, \qquad 0 \le r \le R \tag{6}$$

Noting that the gradient of the net radiant heat flux is represented by[16]

$$\frac{1}{r}\frac{d}{dr}[rq(r)] = (1 - \omega)[4\pi I_b(T) - G(r)] \tag{7}$$

we deduce immediately that Eqs. (5) and (6) can be substituted into Eq. (7) and then integrated. Using the condition that $q(0) = 0$, these operations yield

$$q(r) = (1 - \omega)\left[4\pi \sum_{j=0}^{J} A_j \frac{r^{2j+1}}{2j+2} - \sum_{i=0}^{I} B_i \frac{r^{2i+1}}{2i+2}\right] \tag{8}$$

Thus, it is possible to express the net heat flux in terms of the unknown expansion coefficients B_i. To solve for these expansion coefficients, we substitute the Eqs. (5), (6), and (8) into Eq. (1) to obtain

$$\sum_{i=0}^{I} B_i \left\{r^{2i} - \frac{\omega}{4\pi}\left[D_{i,0}(r) - \frac{a(1-\omega)}{2i+2} D_{i,1}(r)\right]\right\}$$
$$= (1 - \omega)\sum_{j=0}^{J} A_j \left[D_{j,0}(r) + \frac{a\omega}{2j+2} D_{j,1}(r)\right] \tag{9}$$

where the integrals $D_{m,n}(r)$ are defined in the Appendix and evaluated efficiently using numerical integration and recursion relations. To construct a system of $I + 1$ linear algebraic equations for the $I + 1$ unknown expansion coefficients B_i, we use the well established collocation method[17] to obtain

$$\sum_{i=0}^{I} B_i \left\{r_k^{2i} - \frac{\omega}{4\pi}\left[D_{i,0}(r_k) - \frac{a(1-\omega)}{2i+2} D_{i,1}(r_k)\right]\right\}$$
$$= (1 - \omega)\sum_{j=0}^{J} A_j \left[D_{j,0}(r_k) + \frac{a\omega}{2j+2} D_{j,1}(r_k)\right]$$
$$k = 0,1,\ldots,I, \tag{10}$$

where the collocation points r_k are chosen as the zeros of the Chebyshev polynomials shifted onto the interval $[0,R]$ given by

$$r_k = \frac{R}{2}\left[1 + \cos\left(\frac{2k+1}{2(I+1)}\pi\right)\right]$$
$$k = 0,1,\ldots,I \tag{11}$$

Once a solution has been obtained to the expansion coefficients, the incident radiation and the net radiant heat fluxes are, respectively, computed from

$$G(r) = \frac{\omega}{4\pi}\sum_{i=0}^{I} B_i \left[D_{i,0}(r) - \frac{a(1-\omega)}{2i+2} D_{i,1}(r)\right]$$
$$+ (1 - \omega)\sum_{j=0}^{J} A_j \left[D_{j,0}(r) + \frac{a\omega}{2j+2} D_{j,1}(r)\right] \tag{12}$$

$$q(r) = \frac{\omega}{4\pi}\sum_{i=0}^{I} B_i \left[E_{i,0}(r) - \frac{a(1-\omega)}{2i+2} E_{i,1}(r)\right]$$
$$+ (1 - \omega)\sum_{j=0}^{J} A_j \left[E_{j,0}(r) + \frac{a\omega}{2j+2} E_{j,1}(r)\right] \tag{13}$$

where the functions $E_{m,n}(r)$ are defined in the Appendix. It should be noted that for a given order of the expansion more accurate results to the incident radiation and the net radiant heat flux are obtainable using Eqs. (12) and (13), respectively, than using Eqs. (6) and (8). The reason for the improved accuracy is that the integral expressions contain *exact* information, whereas the power series can be viewed as a curve fit.

Using the development of Ref. 15, the expression for the exit distribution of the radiation intensity is constructed as

$$I(R,\theta,\phi) = \int_0^{2R\cos\phi}\left[(1 - \omega)I_b[r(t)] + \frac{\omega}{4\pi}\{G[r(t)]\right.$$
$$\left. + aq[r(t)]\sin\theta(R\cos\phi - t)/r(t)\}\right]$$
$$\times \exp(-t\csc\theta)\csc\theta\, dt$$
$$0 < \theta \le \frac{\pi}{2}, \qquad 0 \le \phi < \frac{\pi}{2} \tag{14}$$

where

$$r(t) = [R^2 + t^2 - 2Rt\cos\phi]^{1/2} \tag{15}$$

Using the series expression for the Planck function, incident radiation and the net radiant heat flux, we obtain

$$I(R,\theta,\phi) = (1 - \omega)\sum_{j=0}^{J} A_j[X_j(R,\theta,\phi) + a\omega Y_j(R,\theta,\phi)]$$
$$+ \frac{\omega}{4\pi}\sum_{i=0}^{I} B_i[X_i(R,\theta,\phi) - a(1 - \omega)Y_i(R,\theta,\phi)]$$
$$0 < \theta \le \frac{\pi}{2}, \qquad 0 \le \phi < \frac{\pi}{2} \tag{16}$$

where the functions $X_i(R,\theta,\phi)$ and $Y_i(R,\theta,\phi)$ are defined in the Appendix. Here we consider the angular interval on (θ,ϕ) in Eq. (14) due to symmetry.

Emissivity of an Isothermal Plume

The IR signature of a plume from a propulsive device is an important source of information for the analysis of the combustion processes. To measure such IR signatures, the detector is located far from the plume and, thus, most likely sees a large portion of the plume. To determine the amount of radiant energy that is incident on a detector, excluding absorption losses through the intervening atmosphere, we consider the model shown in Fig. 2. The radiant energy leaving the surface element dA_1 (on the plume) that strikes the surface element dA_2 (on the detector) is

$$d^2Q_{1\lambda} = I_\lambda(R,\theta,\phi)\cos\beta_1 dA_1 d\Omega_{12} \tag{17}$$

where $d\Omega_{12}$ is the solid angle under which an observer at dA_1 sees the surface element dA_2. To obtain the contribution from all elemental surface areas, we integrate the above expression over the surface A_1 in view of dA_2 and introduce a plume emissivity defined as

$$\varepsilon_\lambda = \frac{\int_{A_1} I_\lambda(R,\theta,\phi)\cos\beta_1 d\Omega_{12} dA_1}{\int_{A_1} I_{b\lambda}(T_m)\cos\beta_1 d\Omega_{12} dA_1} \tag{18}$$

That is, the spectral plume emissivity is defined as the ratio of the spectral radiative flux to the blackbody spectral flux as viewed near the detector. We now limit the analysis to broad-

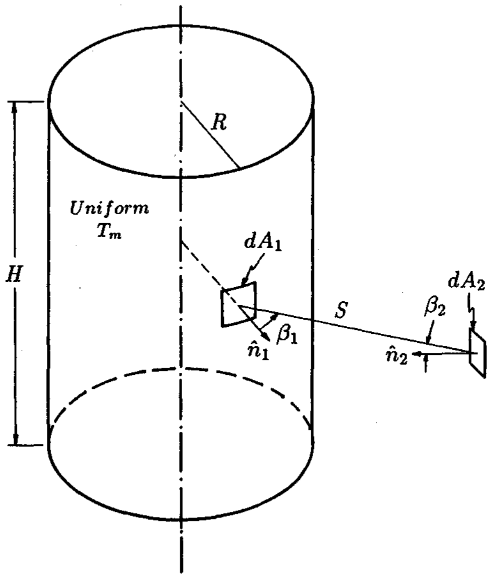

Fig. 2 Schematic of model and coordinates used in the definition of the plume emissivity.

side viewing, which permits us to evaluate the polar angle $\theta = \pi/2$, and replace the angular variable of integration dA_1 with $dA_1 = R_{ph}H\, d\phi$ and $\cos\beta_1 = \cos\phi$. That is, $S >> R_{ph}$ and $S >> H$. The use of these assumptions yields the following expression of the spectral emissivity:

$$\varepsilon_\lambda = \int_0^{\pi/2} \frac{I_\lambda\left(R, \frac{\pi}{2}, \phi\right)}{I_{b\lambda}(T_m)} \cos\phi\, d\phi \tag{19a}$$

with

$$\int_{A_1} I_{b\lambda}(T_m)\cos\beta_1 d\Omega_{12} dA_1 \simeq I_{b\lambda}(T_m) 2R_{ph} H d\Omega_{12} \tag{19b}$$

$$d\Omega_{12} = \cos\beta_2 dA_2/S^2 \tag{19c}$$

With such definition of the spectral emissivity, we deduce that the bounds are $0 \le \varepsilon_\lambda \le 1$ and that the radiant flux intercepted by the detector is computed from

$$dQ_{1\lambda} = \varepsilon_\lambda 2R_{ph} H I_{b\lambda}(T_m) d\Omega_{12} \tag{20}$$

Because the plume is assumed isothermal, the explicit expression for the spectral emissivity becomes

$$\varepsilon_\lambda = (1-\omega)[X_0^* + a\omega Y_0^*] + \frac{\omega}{4\pi}\sum_{i=0}^{I} B_i[X_i^* - a(1-\omega)Y_i^*] \tag{21}$$

where the functions X_i^* and Y_i^*, $i = 0,1,\ldots,I$, are defined in the Appendix and the constants $A_0 = 1$ and $A_j = 0$, $j = 1,2,\ldots,J$. According to our definition, the behavior of the plume emissivity of an absorbing-emitting plume ($\omega = 0$) is represented by the behavior of X_0^*.

The spectral plume emissivity depends on several parameters, including the spectral optical radius, the coefficient for linear-anisotropic scattering and single scattering albedo. To deduce cases for which the gaseous radiation is of importance, or whether the gaseous radiation can be enhanced or inhibited, it is convenient to introduce an interaction parameter C defined as[12]

$$\varepsilon_\lambda = \varepsilon_{\lambda,p} + C\varepsilon_{\lambda,g} \tag{22}$$

where $\varepsilon_{\lambda,p}$ denotes the plume emissivity due to the presence of a (absorbing, emitting and scattering) particulate phase acting alone, whereas $\varepsilon_{\lambda,g}$ represents the plume emissivity due to the (absorbing and emitting) gases acting alone. The definition of such a gas-particulate phase interaction parameter leads to the interpretation that for $C > 1$ the particulate phase enhances the gas emissivity, for $C = 1$ the particulate phase is transparent to the gaseous emission, and for $C < 1$ the particulate phase shields the gaseous emission. If the bounds on C are such that $0 \le C \le 1$, then one may interpret the interaction parameter as an effective transmissivity of the particulate matter. In a related study in plane-parallel media,[12] it has been determined that the normal total emittance of water vapor has been enhanced by up to 95% by the presence of purely isotropically scattering particles. However, a rigorous analysis by Goodwin and Ebert[11] has revealed that the enhancement of the spectral hemispherical emittance of a plane-parallel medium containing isotropically scattering particles and participating gases is very small (maximum 3.2%); that is, the particulate phase shields the gaseous emission very effectively in most cases. In addition, an interaction parameter integrated over all wavelengths could also be defined, but for the purpose of IR signature studies a spectral quantity is of primary interest.

Discussion of Results

To illustrate the application of the previous analysis and to perform a comprehensive parameter survey, we note that there are three independent variables (r,θ,ϕ), two radiative properties of interest (a,ω), and the optical radius R. Thus, there are six parameters to consider in the calculations of the quantities of interest: the radiation intensity $I(R,\theta,\phi)$, the net radiant heat flux $q(R)$ (or the hemispherical emissivity ε_H) at the edge of the plume, and the previously introduced spectral plume emissivity. Here we include the hemispherical emissivity as a variable of interest, because values of this quantity may be of interest to those workers who are involved with modeling of the overall conservation of energy of the plume.

Convergence of Solution

We consider the situation of an isothermal cylinder. Assuming that the expansion coefficient to the temperature $A_0 = I_b(T) = 1$, we show in Table 1 the convergence of the hemispherical emissivity ε_H for $R = 5$ with the different values of $a = -0.99$, 0, and 0.99, and $\omega = 0.1$, 0.5, 0.9, and 0.99. The spectral hemispherical emissivity is defined in the usual manner as

$$\varepsilon_H = q_\lambda(R)/\pi I_{b\lambda}(T) \tag{23}$$

Inspection of the results in this table reveals that results of sufficient accuracy for most engineering applications are obtainable. For example, four terms in the expansion ($I = 3$) yield results that agree quite well with the much higher order expansions. It seems that three- or four-digit accuracy in the computed solution is not necessary in order to perform comprehensive plume studies, because the radiative properties are in general difficult to determine for the particulate matter contained in typical rocket plumes. On the other hand, the high accuracy is of utmost importance if an attempt is made to invert the Eq. (14) for the determination of radiative properties or a reconstruction of the temperature profile or both based on experimental data.[10] Finally, we note that the presence of strongly anisotropic scattering particles ($a = \pm 0.99$) affects the results at the most about 10% compared to the isotropically scattering case. However, the emissivity is strongly dependent on the single scattering albedo ω, indicating that the values of the scattering coefficient σ_s is much more important than the scattering phase function $P(\Theta)$ in estimating the radiative heat transfer from plumes. In addition, the results obtained using this method in the case of isotropic scat-

Table 1 Effects of single scattering albedo ω and anisotropy factor *a* on the emissivity of an absorbing, emitting, anisotropically scattering cylindrical plume of optical radius *R* = 5

		Emissivity ε_H		
ω	l	$a = -0.99$	$a = 0$	$a = 0.99$
	0	0.9887	0.9906	0.9926
	1	0.9681	0.9770	0.9861
	2	0.9612	0.9698	0.9786
0.1	3	0.9603	0.9680	0.9759
	5	0.9603	0.9676	0.9752
	10	0.9604	0.9677	0.9752
	14	0.9604	0.9677	0.9752
	0	0.9702	0.9770	0.9860
	1	0.8376	0.8728	0.9129
	2	0.7970	0.8332	0.8745
0.5	3	0.7927	0.8264	0.8657
	5	0.7928	0.8253	0.8636
	10	0.7934	0.8258	0.8639
	13	0.7935	0.8258	0.8640
	0	0.8797	0.8699	0.8542
	1	0.4531	0.4791	0.5130
	2	0.4116	0.4434	0.4836
0.9	3	0.4089	0.4396	0.4791
	5	0.4088	0.4393	0.4785
	10	0.4094	0.4398	0.4790
	12	0.4095	0.4399	0.4790
	0	0.4434	0.3895	0.3218
	1	0.0902	0.0921	0.0942
	2	0.0857	0.0879	0.0902
0.99	3	0.0850	0.0871	0.0894
	5	0.0850	0.0871	0.0894
	10	0.0852	0.0873	0.0896
	12	0.0852	0.0873	0.0896

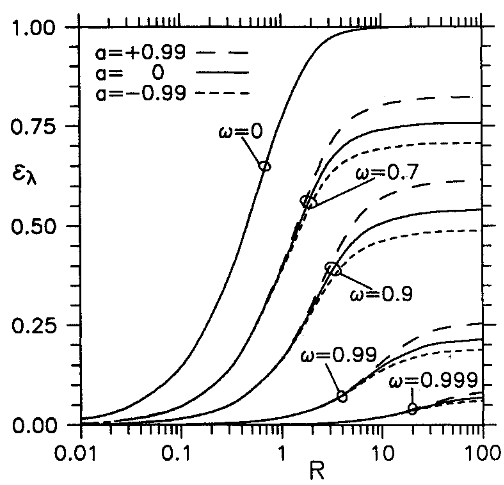

Fig. 3 Effects of single scattering albedo ω, linear scattering coefficient *a*, and optical radius *R* on the plume emissivity ε_λ.

tering are in excellent agreement with the results of Refs. 17 and 18.

Spectral Plume Emissivity ε_λ

The spectral plume emissivity, defined by Eq. (19a), is shown in Fig. 3 for the different values of albedo ω = 0, 0.7, 0.9, 0.99, and 0.999, and linear-anisotropic scattering coefficients a = −0.99, 0, and 0.99. Inspection of Fig. 3 reveals that the spectral plume emissivity never exceeds the value of unity. The reason for this is directly related to the fact that the exit intensity distribution never exceeds the one corresponding to blackbody emission, as discussed in the following sections. Furthermore, near blackbody behavior ($\varepsilon_\lambda \geq 0.99$) requires spectral optical dimensions larger than about five and occurs only if the plume contains purely absorbing constituents. For example, the 4.3 μm band of CO_2 is a strong candidate for exhibiting blackbody behavior near its band center.[19] It is also clear that the single scattering albedo affects the plume emissivity the most, and the effects of the linear anisotropic scattering is negligible for spectral optical dimensions of $R \leq 1$.

Enhancement Factor C

The calculations of the enhancement factor C is shown in Figs. 4–8 for the different values of albedo due to particles only ω_p = 0, 0.7, 0.9, and 1, linear scattering coefficient a = −0.99, 0, and 0.99, and the respective values of the optical radius due to gaseous species only R_g = 0.1, 0.3, 1, 3, and 10. It should be noted that the enhancement factor could be estimated using the results in Fig. 3, and the cases for which $\omega_p = 1$ require $\varepsilon_{\lambda,p} = 0$ in Eq. (22). Examination of Figs. 4 and 5 reveals that the gaseous emission is enhanced if ω_p = 1 for the optical radii R_g = 0.1 and 0.3, respectively; a small enhancement for the forward scattering case is also observed in Fig. 6. That is, the presence of purely scattering particles enhances the gaseous emission as a result of redirection of the emitted radiant energy. The maximum value of the enhancement factor is approximately 1.2, which is obtained in the case of a forward scattering phase function with a = 0.99; this value of the interaction parameter should not be interpreted as the maximum possible value. We should also note that Al_2O_3 particles in a solid phase are nearly purely scattering (ω_p = 1); however, if there exist impurities on the

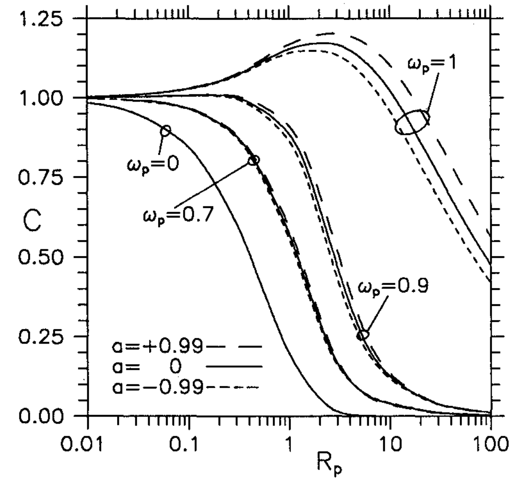

Fig. 4 Effects of particle single scattering albedo ω_p, linear scattering coefficient *a*, and particle optical radius R_p on the enhancement factor *C*; R_g = 0.1.

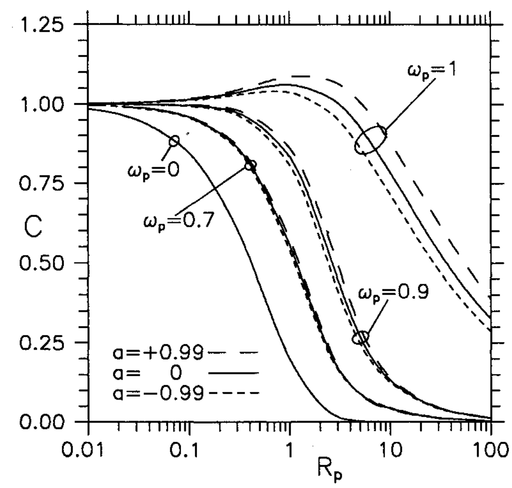

Fig. 5 Effects of particle single scattering albedo ω_p, linear scattering coefficient *a*, and particle optical radius R_p on the enhancement factor *C*; R_g = 0.3.

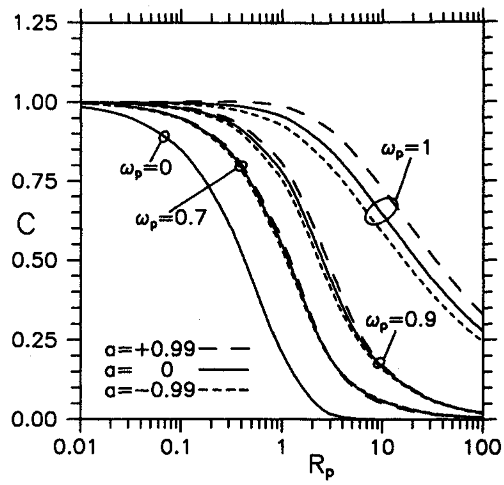

Fig. 6 Effects of particle single scattering albedo ω_p, linear scattering coefficient a, and particle optical radius R_p on the enhancement factor C; R_g = 1.

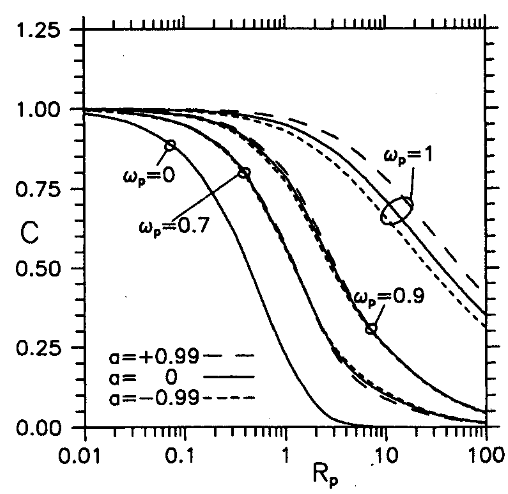

Fig. 7 Effects of particle single scattering albedo ω_p, linear scattering coefficient a, and particle optical radius R_p on the enhancement factor C; R_g = 3.

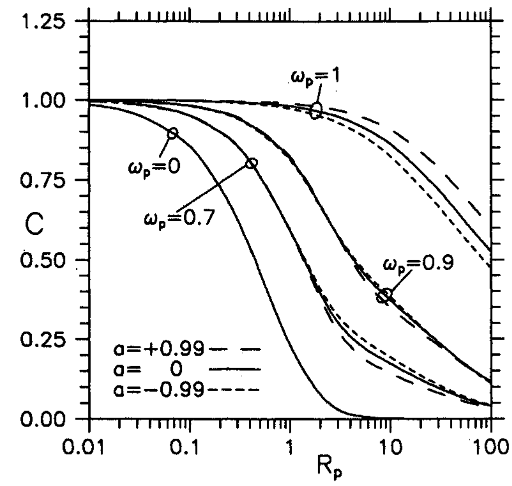

Fig. 8 Effects of particle single scattering albedo ω_p, linear scattering coefficient a, and particle optical radius R_p on the enhancement factor C; R_g = 10.

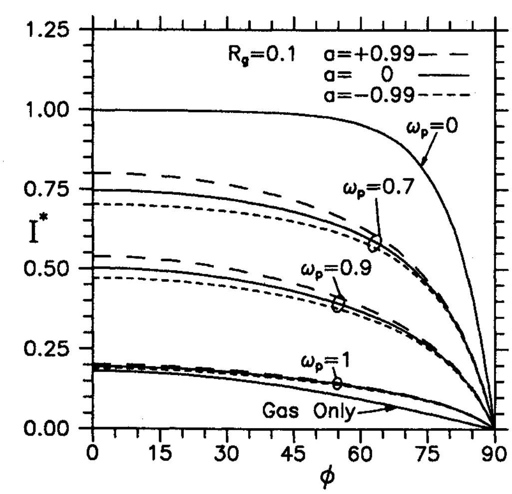

Fig. 9 The normalized exit intensity distribution $I^*(R, \pi/2, \phi)$ of an isothermal plume. The particle optical radius R_p = 3, except for the case of gas only; R_g = 0.1.

surface of the particles, such as soot and binder materials, or if particles are in a molten state,then the absorption cross section is significant ($\omega_p < 1$).[3,4,7] The enhancement effect is not present for the larger values of the gaseous optical radii R_g = 3 and 10 due to the strong self absorption of the emitted radiant energy. Several additional interesting effects are observed in these figures.

First, the enhancement factor in the limit of no particles ($R_p \rightarrow 0$) in all cases approaches unity. This indicates that the enhancement factor is weakly dependent on the particulate matter present within the plume; in other words, the interaction between the particles and the participating gases is weak. In view of the alternative interpretation of C as a transmissivity of the gaseous emission through the particle phase, we deduce that this transmissivity is near unity. In addition, we observe that in this limit the total plume emissivity is a linear combination of the particle and gaseous emissivities.

Second, the enhancement factor in all cases decreases in an exponential-like manner to very small values as the optical radii due to particles become very large ($R_p \rightarrow \infty$). This behavior is expected because the mathematical formulation is first-order and that large particle optical radii shield the gaseous emission very effectively. Thus, for such large values of the particle optical radii, the effect of the type of scattering (forward, isotropic, or backward) on the enhancement factor is insignificant and the curves for the different linear scattering coefficients (a) of a given single scattering albedo coalesce into a single curve. The curves for the different single scattering albedos also tend to coalesce into a single curve, which is most clearly shown in Fig. 8. The reason for this coalescence is directly related to the diminishing role of the gaseous optical radii. The increase in the gaseous optical depth, from R_g = 0.1 in Fig. 4 to R_g = 10 in Fig. 8, also reveals that the enhancement factor more slowly diminishes to a very small value as R_p increases.

Finally, a close inspection of Fig. 7 for ω_p = 0.7 and Fig. 8 for ω_p = 0.7 and 0.9 reveals that the enhancement factor is lower for forward scattering than it is for backward scattering over certain ranges of the particle optical radii. Because this effect is only observed for larger values of the gaseous optical radii (R_g = 3 and 10), it is related to the radiative behavior of the outer layer of the cylinder. Namely, radiant energy emitted within this outer layer in directions toward the edge is likely to escape from the plume because scattering by the layer may not be important; however, radiant energy that is emitted within this layer in directions toward the interior of the plume is absorbed if scattering is highly forward, but it may emerge (reflect out) through the edge of the plume if scattering is highly backward. A similar effect would be obtained if an opaque shield is placed concentric with the plume a few optical units from the edge: coating the shield with a highly absorbing material would simulate forward scattering, whereas coating the shield with a highly reflecting material would simulate backward scattering.

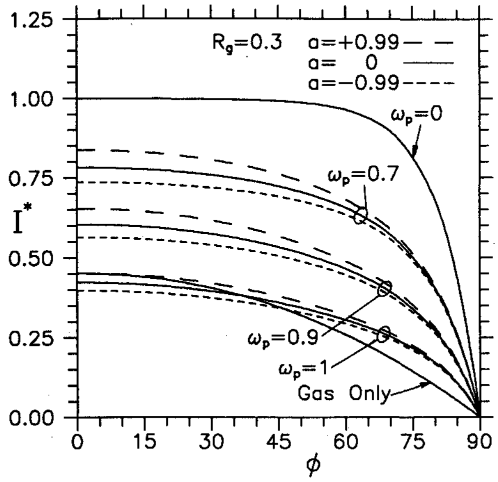

Fig. 10 The normalized exit intensity distribution $I^*(R, \pi/2, \phi)$ of an isothermal plume. The particle optical radius $R_p = 3$, except for the case of gas only; $R_g = 0.3$.

Exit Intensity Distribution $I^*(R,\theta,\phi)$

To examine further how the enhancement of the purely scattering particles manifests itself, we show in Figs. 9 and 10 the exit distribution of the normalized intensity $I^*(R,\theta,\phi) = I(R,\theta,\phi)/I_b(T)$ for $R_g = 0.1$ and $R_g = 0.3$, respectively, with $R_p = 3$ and $\theta = \pi/2$. Here, we have chosen to illustrate the results for $R_p = 3$, because the enhancement effects seem to reach their maximum for the considered combinations of the particle and gaseous optical radii (see Fig. 3). The "gas only" case represents the results for a plume containing gases only. Examination of these figures reveals that the enhancement of the gaseous emission ($C > 1$) is due to an increase in the intensities whose directions are confined to the edge of the plume. That is, intensities whose directions are through the center of the plume ($\phi \simeq 0$) are almost unchanged, whereas intensities whose directions are along the plume edge ($\phi > \simeq 30$) have increased. We also note that pure blackbody behavior is only exhibited through the center of the plume and if the particles are nonscattering ($\omega_p = 0$).

Summary and Conclusions

An analysis of radiative transfer in absorbing, emitting, linear-anisotropic scattering, one-dimensional axisymmetric media has been presented. The spectral solution to this problem has been obtained by using a power series representation of the incident radiation $G(r)$ and medium temperature $T(r)$, which are used to obtain a direct elimination of the spectral radiative heat flux from the problem. As the medium temperature is assumed known (from overall conservation of energy), the expansion coefficients to the incident radiation is solved using a standard collocation strategy. The application of the method for the determination of possible enhancement effects of the gaseous emission has led to the following conclusions:

1) For the considered optical radii of particles and gaseous constituents, and particle scattering properties, a slight ($\simeq$20%) enhancement of the plume emissivity has been predicted. Because this enhancement is observed for the smaller gaseous optical radii, the effect is likely to occur in the band wings of the radiatively active molecules, such as CO_2 or H_2O.[19]

2) Particulate matter whose scattering albedo $\omega_p \leq 0.9$ does not enhance the gaseous emission ($C \leq 1$).

3) The enhancement effect is due to an increase in the intensities whose directions are specified by the larger values of the azimuth angle ϕ.

4) The polynomial expansion technique yields results that are highly accurate. Low order expansions (two or three terms) are of sufficient accuracy for most engineering applications.

Appendix

We define the following integrals:

$$D_{j,n}(r) = \int_0^1 D_{j,n}(r,\mu)\mu^{n-2}\,d\mu \tag{A1}$$

$$E_{j,n}(r) = \int_0^1 E_{j,n}(r,\mu)\mu^{n-2}\,d\mu \tag{A2}$$

where the integrals $D_{j,n}(r,\mu)$ and $E_{j,n}(r,\mu)$ are defined by

$$D_{j,n}(r,\mu) = \int_0^r K_0(r/\mu)I_n(x/\mu)x^{2j+n}x\,dx + \int_r^R I_0(r/\mu)K_n(x/\mu)(-x)^{2j+n}x\,dx \tag{A3}$$

$$E_{j,n}(r,\mu) = \int_0^r K_1(r/\mu)I_n(x/\mu)x^{2j+n}x\,dx - \int_r^R I_1(r/\mu)K_n(x/\mu)(-x)^{2j+n}x\,dx, \tag{A4}$$

where x is the dummy variable of integration. We assume that the integrals in Eqs. (A1) and (A2) are evaluated numerically, that is, we desire the functions $D_{j,n}(r,\mu)$ and $E_{j,n}(r,\mu)$ at a set of quadrature points. Integration by parts yields the following:

$$D_{j,0}(r,\mu) = \mu^2r^{2j} - I_0(r/\mu)[\mu R^{2j+1}K_1(R/\mu) + 2j\mu^2R^{2j}K_0(R/\mu)] + 4j^2\mu^2D_{j-1,0}(r,\mu) \tag{A5}$$

$$D_{j,1}(r,\mu) = \mu R^{2j+2}K_0(R/\mu)I_0(r/\mu) - \mu(2j+2)D_{j,0}(r,\mu) \tag{A6}$$

$$E_{j,0}(r,\mu) = -2j\mu^3R^{2j-1} + I_1(r/\mu)[\mu R^{2j+1}K_1(R/\mu) + 2j\mu^2R^{2j}K_0(R/\mu)] + 4j^2\mu^2E_{j-1,0}(r,\mu) \tag{A7}$$

$$E_{j,1}(r,\mu) = \mu^2r^{2j+1} - \mu R^{2j+2}K_0(R/\mu)I_1(r,\mu) - \mu(2j+2)E_{j,0}(r,\mu) \tag{A8}$$

The functions used in calculating the exit intensity distribution are defined by

$$X_i(R,\theta,\phi) = \int_0^{2R\cos\phi} \csc\theta\exp(-x\csc\theta)(R^2 + x^2 - 2Rx\cos\phi)^i\,dx \tag{A9}$$

$$Y_i(R,\theta,\phi) = \int_0^{2R\cos\phi} \exp(-x\csc\theta)(R^2 + x^2 - 2Rx\cos\phi)^i(R\cos\phi - x)dx/(2i+2) \tag{A10}$$

These integrals are evaluated using recurrence relations. It is readily shown that an integration by parts yields

$$X_i(R,\theta,\phi) = R^{2i}[1 - \exp(-2R\csc\theta\cos\phi)] - 4i^2Y_{i-1}(R,\theta,\phi) \tag{A11a}$$

$$(2i+2)Y_i(R,\theta,\phi) = R^{2i+1}\sin\theta\cos\phi \cdot [1 + \exp(-2R\csc\theta\cos\phi)] - (2i+1)\sin^2\theta \cdot X_i(R,\theta,\phi) + 2iR^2\sin^2\theta\sin^2\phi X_{i-1}(R,\theta,\phi) \tag{A11b}$$

We note that these recurrence relations are self-starting. The functions X_i^* and Y_i^* are defined by

$$X_i^* = \int_0^{\pi/2} X_i\left(R, \frac{\pi}{2}, \phi\right) \cos\phi \, d\phi \tag{A12a}$$

$$Y_i^* = \int_0^{\pi/2} Y_i\left(R, \frac{\pi}{2}, \phi\right) \cos\phi \, d\phi \tag{A12b}$$

These integrals are determined using a 40-point Gauss-Legendre quadrature.

Acknowledgment

The partial support received from the Office of Naval Research, Grant N00014-86-K-0468, is gratefully acknowledged.

References

[1]Lyons, R. B., Wormhoudt, J., and Gruninger, J., "Scattering of Radiation by Particles in Low-Altitudes Plumes," *Journal of Spacecraft and Rockets*, Vol. 20, No. 2, 1983, pp. 189–192.

[2]Ludwig, C. B., Malkmus, W., Freeman, G. N., Slack, M., and Reed, R., "A Theoretical Model for Absorbing, Emitting, and Scattering Plume Radiation," *Progress in Astronautics and Aeronautics: Spacecraft Radiative Transfer and Control*, Vol. 83, edited by T. E. Horton, AIAA, New York, 1982, pp. 111–127.

[3]Nelson, H. F., "Influences of Particulates on Infrared Emission from Tactical Rocket Exhausts," *Journal of Spacecraft and Rockets*, Vol. 21, No. 5, 1984, pp. 425–432.

[4]Nelson, H. F., "Influence of Scattering on Infrared Signatures of Rocket Plumes," *Journal of Spacecraft and Rockets*, Vol. 21, No. 5, 1984, pp. 508–510.

[5]Nelson, H. F., and Tucker, E. O., "Boron Slurry-Fueled Jet Engine Exhaust Plume Infrared Signatures," *Journal of Spacecraft and Rockets*, Vol. 23, No. 5, 1986, pp. 527–533.

[6]Stockham, L. W., and Love, T. J., "Radiative Transfer from a Cylindrical Cloud of Particles," *AIAA Journal*, Vol. 6, No. 10, 1968, pp. 1935–1940.

[7]Edwards, D. K., Sakurai, Y., and Babikian, D. S., "A Two-Particle Model for Rocket Plume Radiation," *Journal of Thermophysics and Heat Transfer*, Vol. 1, No. 1, 1987, pp. 13–20.

[8]Bobco, R. P., "Radiation from Conical Surfaces with Nonuniform Radiosity," *AIAA Journal*, Vol. 4, No. 3, 1966, pp. 544–546.

[9]Young, S. J., "Retrieval of Flow-Field Concentration in Low-Visibility Propellant Rocket Exhaust Plumes," AIAA Paper 81-1054 presented at AIAA Sixteenth Thermophysics Conf., Palo Alto, CA, June 23–25, 1981.

[10]Ho, C.-H., and Özişik, M. N., "Inverse Radiation Problems in Inhomogeneous Media," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 40, No. 5, 1988, pp. 553–560.

[11]Goodwin, D. G., and Ebert, J. L., "Rigorous Bounds on the Radiative Interaction between Real Gases and Scattering Particles," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 37, No. 5, 1987, pp. 501–508.

[12]Skocypec, R. D., and Buckius, R. O., "Total Hemispherical Emittances for CO_2 or H_2O Including Particulate Scattering," *International Journal of Heat and Mass Transfer*, Vol. 27, No. 1, 1984, pp. 1–13.

[13]Buckius, R. O., "The Effect of Molecular Gas Absorption on Radiative Heat Transfer With Scattering," *Journal of Heat Transfer*, Vol. 104, No. 4, 1982, pp. 580–586.

[14]Thynell, S. T., "Radiation due to CO_2 or H_2O and Particles in Cylindrical Media," *Journal of Thermophysics and Heat Transfer*, Vol. 4, No. 4, 1990, pp. 436–445.

[15]Thynell, S. T., "The Integral Form of the Equation of Transfer in Finite Two-Dimensional Cylindrical Media," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 42, No. 2, 1989, pp. 117–136.

[16]Özişik, M. N., *Radiative Transfer*, John Wiley, New York, 1973.

[17]Thynell, S. T., and Özişik, M. N., "Radiation Transfer in an Absorbing, Emitting, Isotropically Scattering, Homogeneous Cylindrical Media," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 38, No. 6, 1987, pp. 413–426.

[18]Siewert, C. E., and Thomas, J. R., Jr., "Radiative Transfer Calculations in Spheres and Cylinders," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 34, No. 1, 1985, pp. 59–64.

[19]Edwards, D. K., "Molecular Gas Band Radiation," *Advances in Heat Transfer*, Vol. 12, edited by T. F. Irvine, Jr. and J. P. Hartnett, Academic Press, New York, 1976, pp. 115–193.